

**EXOTIC GALILEAN SYMMETRY AND THE HALL EFFECT \***

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The “Laughlin” picture of the Fractional Quantum Hall effect can be derived using the “exotic” model based on the two-fold centrally-extended planar Galilei group. When coupled to a planar magnetic field of critical strength determined by the extension parameters, the system becomes singular, and “Faddeev-Jackiw” reduction yields the “Chern-Simons” mechanics of Dunne, Jackiw, and Trugenberger. The reduced system moves according to the Hall law.

**1. Introduction**

In his seminal paper <sup>1</sup> Laughlin argued that the Fractional Quantum Hall Effect <sup>2</sup> could be explained as condensation into a collective ground state, represented by the lowest-Landau-level wave functions

$$f(z)e^{-|z|^2/4}, \quad (1)$$

where the complex  $N$ -vector  $z$  denotes the positions of  $N$  polarized electrons in the plane;  $f(z)$  is analytic. The fundamental operators are  $\widehat{z}f = zf$ , and  $\widehat{\bar{z}}f = 2\partial_z f$  satisfy  $[\widehat{z}, \widehat{\bar{z}}] = 2$ . The quantum Hamiltonian only involves the potential  $V(\bar{z}, z)$  suitably quantized with the choice of an ordering for the non-commuting operators  $\widehat{\bar{z}}$  and  $\widehat{z}$ .

Our results <sup>3</sup> presented here say that the Laughlin picture can actually be obtained from *first principles*, namely using the two-fold central extension of the planar Galilei group. This latter has been known for some time <sup>4 5</sup>, but has long remained a kind of curiosity, since it had no obvious physical use: for a free particle of mass  $m$ , the extra structure related to the new invariant  $k$  leaves the usual motions unchanged, and only contributes to the conserved quantities <sup>3 5 6</sup>. Let us

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mention that our “exotic” theory is in fact equivalent to Quantum Mechanics in the non-commutative plane <sup>7</sup>, with non-commutative parameter  $\theta = k/m^2$ .

Coupling an “exotic” particle to an electromagnetic field, the two extension parameters,  $k$  and  $m$ , combine with the magnetic field,  $B$ , into an effective mass,  $m^*$ , given by (4); when this latter vanishes, the consistency of the equations of motion requires that the particle obey the Hall law. Interestingly, for  $m^* = 0$ , Hamiltonian reduction <sup>8</sup> yields the “Chern-Simons mechanics” considered before by Dunne, Jackiw and Trugenberger <sup>9</sup>. The reduced theory admits the infinite symmetry of area-preserving diffeomorphisms, found before for the edge currents of the Quantum Hall states <sup>10</sup>.

## 2. Exotic particle in a gauge field

Let us consider the action

$$\int (\vec{p} - \vec{A}) \cdot d\vec{x} - h dt + \frac{\theta}{2} \vec{p} \times d\vec{p}, \quad (2)$$

where  $(V, \vec{A})$  is an electro-magnetic potential, the Hamiltonian being  $h = \vec{p}^2/2m + V$ . The term proportional to the non-commutative parameter  $\theta$  is actually equivalent to the acceleration-dependent Lagrangian of Lukierski et al. <sup>6</sup>. The associated Euler-Lagrange equations read

$$\begin{cases} m^* \dot{x}_i &= p_i - m\theta \varepsilon_{ij} E_j, \\ \dot{p}_i &= E_i + B \varepsilon_{ij} \dot{x}_j, \end{cases} \quad (3)$$

where we have introduced the *effective mass*

$$m^* = m(1 - \theta B). \quad (4)$$

The velocity and momentum are different if  $\theta \neq 0$ . The equations of motions (3) can also be written as

$$\omega_{\alpha\beta} \dot{\xi}_\beta = \frac{\partial h}{\partial \xi_\alpha}, \quad \text{where} \quad (\omega_{\alpha\beta}) = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & B \\ 0 & -1 & -B & 0 \end{pmatrix}. \quad (5)$$

Note that the electric and magnetic fields are otherwise arbitrary solutions of the homogeneous Maxwell equation  $\partial_t B + \varepsilon_{ij} \partial_i E_j = 0$ , which guarantees that the two-form  $\omega = \frac{1}{2} \omega_{\alpha\beta} d\xi_\alpha \wedge d\xi_\beta$  is closed,  $d\omega = 0$ .

When  $m^* \neq 0$ , the determinant  $\det(\omega_{\alpha\beta}) = (1 - \theta B)^2 = (m^*/m)^2$  is nonzero and the matrix  $(\omega_{\alpha\beta})$  in (5) can be inverted. Then the equations of motion (5) (or (3)) take the form  $\dot{\xi}_\alpha = \{\xi_\alpha, h\}$ , with the standard Hamiltonian, but with the new

Poisson bracket  $\{f, g\} = (\omega^{-1})_{\alpha\beta} \partial_\alpha f \partial_\beta g$  which reads, explicitly,

$$\begin{aligned} \{f, g\} = & \frac{m}{m^*} \left[ \frac{\partial f}{\partial \vec{x}} \cdot \frac{\partial g}{\partial \vec{p}} - \frac{\partial g}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{p}} \right. \\ & \left. + \theta \left( \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial g}{\partial x_1} \frac{\partial f}{\partial x_2} \right) + B \left( \frac{\partial f}{\partial p_1} \frac{\partial g}{\partial p_2} - \frac{\partial g}{\partial p_1} \frac{\partial f}{\partial p_2} \right) \right]. \end{aligned} \quad (6)$$

Further insight can be gained when the magnetic field  $B$  is a (positive) nonzero constant, which turns out the most interesting case, and will be henceforth assumed. (The electric field  $E_i = -\partial_i V$  is still arbitrary). Introducing the new coordinates

$$\begin{cases} Q_i = x_i + \frac{1}{B} \left[ 1 - \sqrt{\frac{m^*}{m}} \right] \varepsilon_{ij} p_j, \\ P_i = \sqrt{\frac{m^*}{m}} p_i - \frac{1}{2} B \varepsilon_{ij} Q_j, \end{cases} \quad (7)$$

will allow us to generalize our results in <sup>3</sup> from a constant to any electric field.

Firstly, the Cartan one-form <sup>12</sup> in the action (2) reads simply  $P_i dQ_i - h dt$ , so that the symplectic form on phase space retains the canonical guise,  $\omega = dP_i \wedge dQ_i$ . The price to pay is that the Hamiltonian becomes rather complicated <sup>3</sup>.

The equations of motion (3) are conveniently presented in terms of the new variables  $\vec{Q}$  and the old momenta  $\vec{p}$ , as

$$\begin{cases} \dot{Q}_i = \varepsilon_{ij} \frac{E_j}{B} + \sqrt{\frac{m}{m^*}} \left( \frac{p_i}{m} - \varepsilon_{ij} \frac{E_j}{B} \right), \\ \dot{p}_i = \varepsilon_{ij} B \frac{m}{m^*} \left( \frac{p_j}{m} - \varepsilon_{jk} \frac{E_k}{B} \right). \end{cases} \quad (8)$$

Note that all these expressions diverge when  $m^*$  tends to zero.

When the magnetic field takes the particular value

$$B = B_c = \frac{1}{\theta}, \quad (9)$$

the effective mass (4) vanishes,  $m^* = 0$ , so that  $\det(\omega_{\alpha\beta}) = 0$ , and the system becomes singular. Then the time derivatives  $\dot{\xi}_\alpha$  can no longer be expressed from the variational equations (5), and we have resort to “Faddeev-Jackiw” reduction <sup>8</sup>. In accordance with the Darboux theorem (see, e.g., <sup>12</sup>), the Cartan one-form in (2) can be written, up to an exact term, as

$$\vartheta - h dt, \quad \text{with} \quad \vartheta = (p_i - \frac{1}{2} B_c \varepsilon_{ij} x_j) dx_i + \frac{1}{2} \theta \varepsilon_{ij} p_i dp_j = P_i dQ_i, \quad (10)$$

where the new coordinates read, consistently with (7),

$$Q_i = x_i + \frac{1}{B_c} \varepsilon_{ij} p_j, \quad (11)$$

while the  $P_i = -\frac{1}{2}B_c \varepsilon_{ij} Q_j$  are in fact the rotated coordinates  $Q_i$ . Eliminating the original coordinates  $\vec{x}$  and  $\vec{p}$  using (11), we see that the Cartan one-form reads  $P_i dQ_i - H(\vec{Q}, \vec{p}) dt$ , where  $H(\vec{Q}, \vec{p}) = \vec{p}^2/(2m) + V(\vec{Q}, \vec{p})$ . As the  $p_i$  appear here with no derivatives, they can be eliminated using their equation of motion  $\partial H(\vec{Q}, \vec{p})/\partial \vec{p} = 0$ , i. e., the constraint

$$\frac{p_i}{m} - \frac{\varepsilon_{ij} E_j}{B_c} = 0. \quad (12)$$

A short calculation shows that the reduced Hamiltonian is just the original potential, viewed as a function of the “twisted” coordinates  $\vec{Q}$ , *viz.*

$$H = V(\vec{Q}). \quad (13)$$

This rule is referred to as the “Peierls substitution”<sup>9, 3</sup>. Since  $\partial^2 H/\partial p_i \partial p_j = \delta_{ij}/m$  is already non singular, the reduction stops, and we end up with the reduced Lagrangian

$$L_{\text{red}} = \frac{1}{2\theta} \vec{Q} \times \dot{\vec{Q}} - V(\vec{Q}), \quad (14)$$

supplemented with the Hall constraint (12). The 4-dimensional phase space is hence reduced to 2 dimensions, with  $Q_1$  and  $Q_2$  in (11) as canonical coordinates, and reduced symplectic two-form  $\omega_{\text{red}} = \frac{1}{2}B_c \varepsilon_{ij} dQ_i \wedge dQ_j$  so that the reduced Poisson bracket is

$$\{F, G\}_{\text{red}} = -\frac{1}{B_c} \left( \frac{\partial F}{\partial Q_1} \frac{\partial G}{\partial Q_2} - \frac{\partial G}{\partial Q_1} \frac{\partial F}{\partial Q_2} \right). \quad (15)$$

The twisted coordinates are therefore again non-commuting,

$$\{Q_1, Q_2\}_{\text{red}} = -\theta = -\frac{1}{B_c}. \quad (16)$$

The equations of motion associated with (14), and also consistent with the Hamilton equations  $\dot{Q}_i = \{Q_i, H\}_{\text{red}}$ , are given by

$$\dot{Q}_i = \varepsilon_{ij} \frac{E_j}{B_c}, \quad (17)$$

in accordance with the Hall law (compare (8) with the divergent terms removed).

Putting  $B_c = 1/\theta$ , the Lagrangian (14) becomes formally identical to the one Dunne et al.<sup>9</sup> derived letting the *real* mass go to zero. Note, however, that while  $\vec{Q}$  denotes real position in Ref.<sup>9</sup>, our  $\vec{Q}$  here is the “twisted” expression (11), with the magnetic field frozen at the critical value  $B_c = 1/\theta$ .

### 3. Infinite symmetry

It has been argued<sup>11</sup> that the physical process which yields the Fractional Quantum Hall Effect actually takes place at the boundary of the droplet of the “Hall” liquid: owing to incompressibility, the bulk can not support any density waves, but there are chiral currents at the edge. These latter fall into irreducible representations of the infinite dimensional algebra  $W_{1+\infty}$ <sup>10</sup>, which is the quantum deformation of

$w_\infty$ , the algebra of classical observables which generate the group of area-preserving diffeomorphisms of the plane.

Our reduced model is readily seen to admit  $w_\infty$ , the classical counterpart of  $W_{1+\infty}$ , as symmetry. To see this, let us remember that, as argued by Souriau<sup>12</sup>, and later by Crnkovic and Witten<sup>13</sup>, it is convenient to consider the space of solutions of the equations of motion (Souriau's "*espace des mouvements*" [= space of motions]), denoted by  $\mathcal{M}$ . For a classical mechanical system, this is an abstract substitute for the classical phase space, whose points are the motion curves of the system. The classical dynamics is encoded into the symplectic form  $\Omega$  of  $\mathcal{M}$ . It is then obvious that *any* function  $f(\zeta)$  on  $\mathcal{M}$  is a constant of the motion. (When expressed using the positions, time, and momenta, such a function can look rather complicated). Any such function  $f(\zeta)$  generates a Hamiltonian vectorfield  $Z^\mu$  on  $\mathcal{M}$  through the relation

$$-\partial_\mu f = \Omega_{\mu\nu} Z^\nu. \quad (18)$$

The vector field  $Z^\mu$  generates, at least locally, a 1-parameter group of diffeomorphisms of  $\mathcal{M}$ . All diffeomorphisms of  $\mathcal{M}$  which leave the symplectic form  $\Omega$  invariant form an infinite dimensional group, namely the group of symplectomorphisms of  $\mathcal{M}$ . Any symplectic transformation is a symmetry of the system : it merely permutes the motions curves.

For the reduced system above, the reduced phase space is two dimensional. The space of motions is therefore locally a plane. (Its global structure plainly depends on the details of the dynamics). Now, for any orientable two dimensional manifold, the symplectic form is the area element; it follows that the reduced system admits the group of area-preserving transformations as symmetry.

#### 4. Quantization

Let us conclude our general theory by quantizing the coupled system. Again, owing to the exotic term, the position representation does not exist.

Introducing the complex coordinates

$$\begin{cases} z = \frac{\sqrt{B}}{2}(Q_1 + iQ_2) + \frac{1}{\sqrt{B}}(-iP_1 + P_2) \\ w = \frac{\sqrt{B}}{2}(Q_1 - iQ_2) + \frac{1}{\sqrt{B}}(-iP_1 - P_2) \end{cases} \quad (19)$$

the two-form  $dP_i \wedge dQ_i$  on 4-dimensional unreduced phase space becomes the canonical Kähler two-form of  $\mathbf{C}^2$ , viz  $\omega = (2i)^{-1}(d\bar{z} \wedge dz + d\bar{w} \wedge dw)$ . choosing the antiholomorphic polarization, the "unreduced" quantum Hilbert space, consisting of the "Bargmann-Fock" wave functions

$$\psi(z, \bar{z}, w, \bar{w}) = f(z, w) e^{-\frac{1}{4}(z\bar{z} + w\bar{w})}, \quad (20)$$

where  $f$  is holomorphic in both of its variables. The fundamental quantum opera-

tors,

$$\begin{cases} \widehat{z}f = zf, & \widehat{\bar{z}}f = 2\partial_z f, \\ \widehat{w}f = wf, & \widehat{\bar{w}}f = 2\partial_w f, \end{cases} \quad (21)$$

satisfy the commutation relations  $[\widehat{z}, \widehat{z}] = [\widehat{w}, \widehat{w}] = 2$ , and  $[\widehat{z}, \widehat{w}] = [\widehat{\bar{z}}, \widehat{\bar{w}}] = 0$ . We recognize here the familiar creation and annihilation operators, namely  $a_z^* = z$ ,  $a_w^* = w$ , and  $a_z = \partial_z$ ,  $a_w = \partial_w$ . Using (7), the (complex) momentum  $p = p_1 + ip_2$  and the kinetic part,  $h_0$ , of the Hamiltonian become, respectively,

$$p = -i\sqrt{\frac{mB}{m^*}} \bar{w} \quad \text{and} \quad h_0 = \frac{B}{2m^*} w\bar{w}. \quad (22)$$

For  $m^* \neq 0$  the wave function satisfies the Schrödinger equation  $i\partial_t f = \widehat{h}f$ , with  $\widehat{h} = \widehat{h}_0 + \widehat{V}$ . The quadratic kinetic term here is

$$\widehat{h}_0 = \frac{B}{4m^*} (\widehat{w}\widehat{\bar{w}} + \widehat{\bar{w}}\widehat{w}) = \frac{B}{2m^*} (\widehat{w}\widehat{\bar{w}} + 1). \quad (23)$$

The case when the effective mass tends to zero is conveniently studied in this framework. On the one hand, in the limit  $m^* \rightarrow 0$ , one has

$$z \rightarrow \sqrt{B}Q, \quad w \rightarrow 0, \quad (24)$$

where  $Q = Q_1 + iQ_2$ , cf. (7); the 4-dimensional phase space reduces to the complex plane. On the other hand, from (22) and (21) we deduce that

$$i\sqrt{\frac{m^*}{mB}} \widehat{p} = \widehat{w} = 2\partial_w. \quad (25)$$

The limit  $m^* \rightarrow 0$  is hence enforced, at the quantum level, by requiring that the wave functions be independent of the coordinate  $w$ , i.e.,

$$\partial_w f = 0, \quad (26)$$

yielding the reduced wave functions of the form

$$\Psi(z, \bar{z}) = f(z) e^{-\frac{1}{4}z\bar{z}}, \quad (27)$$

where  $f$  is a holomorphic function of the reduced phase space parametrized by  $z$ . When viewed in the “big” Hilbert space (see (20)), these wave functions belong, by (23), to the lowest Landau level<sup>2 14 3</sup>.

Using the fundamental operators  $\widehat{z}$  and  $\widehat{\bar{z}}$  given in (21), we easily see that the (complex) “physical” position  $x = x_1 + ix_2$  and its quantum counterpart  $\widehat{x}$ , namely

$$x = \frac{1}{\sqrt{B_c}} \left( z + \sqrt{\frac{m}{m^*}} \bar{w} \right), \quad \widehat{x} = \frac{1}{\sqrt{B_c}} \left( z + \sqrt{\frac{m}{m^*}} 2\partial_w \right), \quad (28)$$

manifestly diverge when  $m^* \rightarrow 0$ . *Positing from the outset* the conditions (26) the divergence is suppressed, however, leaving us with the reduced position operators

$$\widehat{x}f = \widehat{Q}f = \frac{1}{\sqrt{B_c}} zf, \quad \widehat{\bar{x}}f = \widehat{\bar{Q}}f = \frac{2}{\sqrt{B_c}} \partial_z f, \quad (29)$$

whose commutator is  $[\widehat{Q}, \widehat{Q}] = 2/B_c$ , cf. (16). In conclusion, we recover the “Laughlin” description (1) of the ground states of the FQHE in <sup>2</sup>. Quantization of the reduced Hamiltonian (which is, indeed, the potential  $V(z, \bar{z})$ ), can be achieved using, for instance, anti-normal ordering <sup>9 14</sup>.

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